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Determination of an Optimal Mesh for a Collocation–Projection Method for Solving Two-Point Boundary Value Problems*

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We consider mesh-point optimization for certain collocation–projection methods for solving boundary value problems (BVP) for ordinary differential equations. Consider the BVP

$$(L_m x)(t) = g(t), \qquad t \in [0, 1],$$
$$x^{(\nu)}(0) = x^{(\nu)}(1) = 0, \qquad \nu = 0, 1,\dots, m/2 - 1,$$

where L_m is an mth order linear differential operator. We assume conditions guaranteeing that there is a unique solution in a given (reproducing kernel) Hilbert space $\mathscr{H}_R$, and that $|(L_m x)(t)| \leqslant M \| x \|_R$, $0 \leqslant t \leqslant 1$, for all $x \in \mathscr{H}_R$, and some M, where $\| \cdot \|_R$ is the norm in $\mathscr{H}_R$. For a given mesh $T_N = \{t_{iN}\}_{i=1}^N$, we let x_N, our approximate solution to the BVP, be that element in $\mathscr{H}_R$ of minimal norm which satisfies the boundary conditions and matches the data on T_N, that is, $(L_m x_N)(t_{iN}) = g(t_{iN})$, $i = 1, 2,\dots, N$. x_N is both a collocation and an orthogonal projection approximation to x, and for certain $\mathscr{H}_R$ equivalent to the Sobolev space $W_2^{(r)}$, x_N is a spline approximant to x. We are interested in choosing the mesh T_N so that $\| x - x_N \|_R$ is as small as possible. The optimal mesh we are after depends on the unknown x. Under certain circumstances a mesh behaving essentially like an optimal mesh can be characterized by a cumulative distribution function F^* on [0, 1], which depends on x. A (nearly) optimal mesh $T_N^* = \{t_{iN}^*\}$ is determined by solving $F^*(t_{iN}^*) = i/(N + 1)$, $i = 1, 2,\dots, N$. F^* has been given by Sacks and Ylvisaker [*Ann. Math. Statist.* **37**, No. 1 (1966)] and Wahba [*Ann. Math. Statist.* 42 (1971); *Ann. Statist.* 2, No. 5 (1974); *J. Approximation Theory* **16** (1976)] under various conditions. In this paper we show how an estimate F_n^* of F^* can be computed from data starting with an arbitrary (nice) mesh with n points. Once F_n^* is obtained, then a new mesh, say $\hat{T}_N = \{\hat{t}_{iN}\}_{i=1}^N$ can be obtained as $F_n^*(\hat{t}_{iN}) = i/(N + 1)$, $i = 1, 2,\dots, N$, and the final approximate solution

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x_N computed using the estimated approximately optimal mesh $\hat{T}_N$. From a different point of view, if data points $g(t_{iN})$ are obtained from an experiment and are expensive to measure, this approach is that of the sequential design of an experiment. Data at a preliminary uniform mesh (design) are obtained, and this data is used to obtain an improved mesh (design). These results apply to more general linear operator equations.

1. Introduction and Preliminaries

Consider the boundary value problem (BVP)

$$\begin{aligned} (L_m x)(t) &= g(t), \qquad t \in [0, 1], \\ x^{(\nu)}(0) = x^{(\nu)}(1) &= 0, \qquad \nu = 0, 1, \ldots, m/2 - 1, \end{aligned} \tag{1.1}$$

where L_m is an mth order differential operator with an m-dimensional null space. We suppose that the domain of L_m is a reproducing kernel Hilbert space (RKHS) which contains the null space of L_m. Letting $\| \cdot \|_R$ be the norm in $\mathscr{H}_R$, we further suppose

(a) $|(L_m x)(t)| \leqslant M \| x \|_R, \quad x \in \mathscr{H}_R, \quad t \in [0, 1],$

(b) $| x^{(\nu)}(i)| \leqslant M \| x \|_R, \quad x \in \mathscr{H}_R, \quad \nu = 0, 1, \ldots, m/2 - 1, \quad i = 0, 1.$

Conditions (a) and (b) are always satisfied, for example, if $\mathscr{H}_R$ is the Sobolev space $W_2^{(r)}$,

$$W_2^{(r)} = \{x\colon x, x', \ldots, x^{(r-1)} \text{ abs. cont.}, x^{(r)} \in \mathscr{L}_2[0, 1]\},$$

with $r > m$. Then by the Riesz representation theorem there exist $\{\eta_t, t \in [0, 1]\}$, and $\{R_{i\nu}, i = 0, 1, \nu = 0, 1, \ldots, m/2 - 1\}$ in $\mathscr{H}_R$ such that

$$\begin{aligned} (L_m x)(t) &= \langle \eta_t, x \rangle_R, \qquad t \in [0, 1], \\ x^{(\nu)}(i) &= \langle R_{i\nu}, x \rangle_R, \qquad i = 0, 1, \quad \nu = 0, 1, \ldots, m/2 - 1. \end{aligned}$$

Given a mesh $T_N = \{t_{iN}\}_{i=1}^N \equiv \{t_i\}_{i=1}^N$, and data $g(t_i)$, $i = 1, 2, \ldots, N$, we take as an approximate solution, that element x_N in $\mathscr{H}_R$ of minimal $\mathscr{H}_R$ norm satisfying the boundary conditions

$$\langle R_{i\nu}, x_N \rangle_R = 0, \qquad i = 0, 1, \quad \nu = 0, 1, \ldots, m/2 - 1$$

and matching the data on the given mesh

$$\langle \eta_{t_i}, x \rangle_R = g(t_i), \qquad i = 1, 2, \ldots, N.$$

We will assume

(c) $\{\eta_t, \ t \in [0, 1], \ R_{i\nu}, \ i = 0, 1, \ \nu = 0, 1, \ldots, m/2 - 1\}$ are linearly independent and span $\mathscr{H}_R$.

Condition (c) can be shown to be satisfied, if, e.g.,

$$L_m = \sum_{j=0}^{m} a_j(t)D^m \tag{1.2}$$

with $\infty > a_m(t) > 0$, and $\mathscr{H}_R = W_2^{(r)}$, $r > m$.

It is not hard to show (see [2]) that conditions (a), (b), and (c) guarantee a unique solution in $\mathscr{H}_R$ to the BVP for any $g \in L_m(\mathscr{H}_R)$, and that x_N is uniquely determined for any mesh and data vector. x_N is both a collocation approximation and an orthogonal projection approximation to x. Letting $V_N = \text{span}\{\eta_{t_1}, ..., \eta_{t_N}\}$, $S_m = \text{span}\{R_{i\nu}, i = 0, 1, \nu = 0, 1, ..., m/2 - 1\}$, x_N is the orthogonal projection of x onto $V_N \oplus S_m$. This type of approximate solution was suggested in [5]. If the reproducing kernel for $\mathscr{H}_R$ is taken, e.g., as

$$R(s, t) = \sum_{i=0}^{r-1} \frac{s^j t^j}{(j!)^2} + \int_0^{\min(s,t)} \frac{(s-u)_+^{r-1}(t-u)_+^{r-1}}{[(r-1)!]^2}\, du, \tag{1.3}$$

then $\mathscr{H}_R$ is topologically equivalent to $W_2^{(r)}$. In this case the $R_{i\nu}$ are polynomials, $\eta_{t_i}(\cdot)$ is a polynomial spline of degree $2r - 1$, possessing $2r - m - 2$ continuous derivatives and a single knot at t_i, and x_N is a polynomial spline of degree $2r - 1$ and continuity class $2r - m - 2$ with knots at the collocation points. To verify this last assertion, let $R_s(\cdot) = R(s, \cdot)$ be the representer of the evaluation functional at s in $\mathscr{H}_R$, then

$$R_{i\nu}(s) = \langle R_{i\nu}, R_s \rangle_R = \frac{\partial^{(\nu)}}{\partial t^{(\nu)}} R(s, t) \Big|_{t=i}, \tag{1.4}$$

$$\eta_{t_i}(s) = \langle \eta_{t_i}, R_s \rangle_R = \sum_{\nu=0}^{m-1} a_\nu(t_i) \frac{\partial^{(\nu)}}{\partial t^{(\nu)}} R(s, t) \Big|_{t=t_i}, \tag{1.5}$$

and it can be checked that the functions of s on the right have the claimed properties. (Details may be found in [2].)

We would like to choose T_N to minimize $\| x - x_N \|_R$. We do not know how to do this, but will do something very close, as follows:

Let $V = \overline{\text{span}}\{\eta_t, t \in [0, 1]\}$. By (c), $\mathscr{H}_R = V \oplus S_m$. Now, S_m is of dimension m. The codimension of V is also m, since $V^\perp$ is of dimension m. This follows since $V^\perp$ is the null space of L_m, since $\langle \eta_t, x \rangle_R = 0$, $t \in [0, 1]$ if and only if $L_m x = 0$. Thus each $x \in \mathscr{H}_R$ has a unique decomposition

$$x = y + z \tag{1.6}$$

with $y \in V$ and $z \in S_m$.

Letting P_S and P_{V_N} be the orthogonal projectors in $\mathscr{H}_R$ onto $S_m \oplus V_N$ and V_N, respectively, we have

$$\begin{aligned} \| x - x_N \|_R &= \| x - P_S x \|_R = \|(y + z) - P_S(y + z)\|_R \\ &\leqslant \| y - P_S y \|_R + \| z - P_S z \|_R \\ &= \| y - P_S y \|_R \\ &\leqslant \| y - P_{V_N} y \|_R . \end{aligned}$$

We will be able to choose T_N with the goal of minimizing $\| y - P_{V_N} y \|_R$.

Let

$$F(t) = \int_0^t f(s)\, ds,$$

where $f > 0$ and $F(1) = 1$, and let $T_N = T_N(f)$ be determined by $T_N = \{t_{iN}\}_{i=1}^N$, as

$$F(t_{iN}) = i/(N + 1), \qquad i = 1, 2,..., N. \tag{1.7}$$

A uniform mesh corresponds to $f(s) = 1$, $s \in [0, 1]$. It is known from [7, 8, 12], that, under some further conditions on y, the large N behavior of $\| y - P_{V_N} y \|_R$ can be described fairly precisely as a function of the mesh cumulative distribution function (c.d.f.) F appearing in (1.7). Furthermore the F, call it F^* which (loosely) minimizes $\| y - P_{V_N} y \|_R$ is known from [7, 8, 12], and, it depends on the unknown y. In this paper we construct, starting from an arbitrary (nice) mesh with n points, and the values of g on this mesh, an approximation F_n^* to F^*. Once F_n^* is obtained, a new mesh $\hat{T}_N = \{t_{iN}\}$, say, can be determined from

$$F_n^*(\hat{t}_{iN}) = i/(N + 1), \qquad i = 1, 2,..., N.$$

Then this new mesh can be used to compute the final approximant x_N.

This technique has a greater generality than BVP's, we indicate its usefulness for more general linear operator equations at the end.

We now proceed to describe the results from [12] that we need. Let $Q(s, t) = \langle \eta_s, \eta_t \rangle_R = (L_m \eta_t)(s) = L_{m(s)} L_{m(t)} R(s, t)$, where $L_{m(s)}$ means L_m applied to a function of s. Let $\mathscr{H}_Q$ be the RKHS with RK Q. There exists an isometric isomorphism between V and $\mathscr{H}_Q$ generated by the correspondence

$$\eta_t \in V \sim Q_t \in \mathscr{H}_Q,$$

where $Q_t(\cdot) = Q(t, \cdot)$ is the representer of evaluation at t in $\mathscr{H}_Q$. To see this note that

$$\langle \eta_s, \eta_t \rangle_R = Q(s, t) = \langle Q_s, Q_t \rangle_Q .$$

Furthermore $y \in V \sim u \in \mathscr{H}_Q$ iff $L_m y = u$. Since $\mathscr{H}_R = V \oplus V^\perp$, where $V^\perp$ is the null space of L_m, $L_m(\mathscr{H}_R) = L_m(V) = \mathscr{H}_Q$. Let P_{T_N} be the orthogonal projector in $\mathscr{H}_Q$ onto span$\{Q_{t_1},..., Q_{t_N}\}$. Then if $y \in V \sim u \in \mathscr{H}_Q$, by the aforementioned isometric isomorphism,

$$\| y - P_{V_N} y \|_R = \| u - P_{T_N} u \|_Q . \tag{1.8}$$

Consider a $u \in \mathscr{H}_Q$ with a representation

$$u(t) = \int_0^1 Q(t, s)\, \rho(s)\, ds \tag{1.9}$$

for some $\rho \in \mathscr{L}_2$. It is for u of the form (1.9) that $\| u - P_{T_N} u \|_Q$ can be described in terms of the mesh c.d.f. We have $y \in V \sim u$ of (1.9) if

$$y(t) = \int_0^1 \eta_t(s)\, \rho(s)\, ds, \tag{1.10}$$

equivalently

$$(L_m y)(t) = \int_0^1 Q(t, s)\, \rho(s)\, ds \equiv u(t). \tag{1.11}$$

(We note that $u \neq g$ unless $S_m = V^\perp$.) We will henceforth assume that y has a representation (1.10). If $\mathscr{H}_R = W_2^{(r)}$, then (1.10) entails that $x \in W_2^{(m+2q)} = W_2^{(2r-m)}$.

It is supposed in [12] (loosley), that $Q(s, t)$ has the continuity properties of a Green's function of a self-adjoint linear differential operator of order $2q$, with

$$\lim_{s \downarrow t} \frac{\partial^{2q-1}}{\partial s^{2q-1}} Q(s, t) - \lim_{s \uparrow t} \frac{\partial^{2q-1}}{\partial s^{2q-1}} Q(s, t) = (-1)^m \alpha(t) \tag{1.12}$$

with α and α' continuous. (More generally the hypotheses of the theorem in [12, Section 2] are being assumed.) If L_m and $\mathscr{H}_R$ are given by (1.2) and (1.3) then $q = r - m$ and $\alpha(t) = 1/a_m^2(t)$. With these assumptions, the linear functionals $N_{j,k} u \to u^{(k)}(t_j)$ are continuous in $\mathscr{H}_Q$ for $k = 0, 1,..., q - 1$. Let $Q_{t_j,k}$ be the representer of $N_{j,k}$ in $\mathscr{H}_Q$, and let P_{q,T_n} be the projection operator in $\mathscr{H}_Q$ onto span$\{Q_{t_j,k}\}_{j=1,k=0}^{n\ \ q-1}$, and let P_{T_n} be the projection operator in $\mathscr{H}_Q$ onto span$\{Q_{t_j}\}_{j=1}^n \equiv \{Q_{t_j,0}\}_{j=1}^n$. Then for $u \in \mathscr{H}_Q$,

$$\inf_{T_{qn}} \| u - P_{q,T_{qn}} u \|_Q \leq \inf_{T_{qn}} \| u - P_{T_{qn}} u \|_Q \leq \inf_{T_n} \| u - P_{q,T_n} u \|_Q , \tag{1.13}$$

where $\inf_{V_k}$ means the infimum is taken over all k-points designs. It is known that if $q = 2$, the right-hand inequality becomes an equality. See [8] for details. The reason for presenting these inequalities is that an exact asymptotic expression is available for $\| u - P_{q,T_n} u \|_Q$, and is given by

THEOREM [11, 12]. *Let f be a strictly positive density, $F(t) = \int_0^t f(u)\, du$, let ρ and α be as in* (1.10) *and* (1.12), *and suppose ρ and α are strictly positive,* *continuous, and possess bounded first derivatives. Let* $T_N = T_N(f) = \{t_{iN}\}_{i=1}^N$, $N = 1, 2, \ldots$ *be determined by*

$$F(t_{iN}) = i/(N+1), \qquad i = 1, 2, \ldots .$$

Then

$$\| u - P_{q,T_N} u \|_Q^2 = \frac{1}{N^{2q}} \frac{(q!)^2}{(2q)!\,(2q+1)!} \int_0^1 \frac{\rho^2(s)\,\alpha(s)}{f^{2q}(s)}\, ds\,(1 + o(1)). \tag{1.14}$$

where $o(1) \to 0$ *as* $N \to \infty$. *Furthermore, by using a Hölder inequality and the fact that* $\int_0^1 f(s)\, ds = 1$, *we have*

THEOREM [11, 12].

$$\int_0^1 \frac{\rho^2(s)\,\alpha(s)}{f^{2q}(s)}\, ds \geqslant \left[\int_0^1 [\rho^2(s)\,\alpha(s)]^{1/2q}\, ds\right]^{2q+1} \tag{1.15}$$

and the lower bound on the right of (1.15) *is achieved if and only if* $f = f^*$ *given by*

$$f^*(s) = \frac{[\rho^2(s)\,\alpha(s)]^{1/(2q+1)}}{\int_0^1 [\rho^2(u)\,\alpha(u)]^{1/(2q+1)}\, du}, \tag{1.16}$$

that is,

$$F^*(t) = \int_0^t [\rho^2(s)\,\alpha(s)]^{1/(2q+1)}\, ds \Big/ \int_0^1 [\rho^2(s)\,\alpha(s)]^{1/(2q+1)}\, ds. \tag{1.17}$$

We remark that all the regularity conditions on ρ and α are probably not necessary but are artifacts of earlier proofs.

The result of this paper is as follows. Given a uniform mesh $\{t_{in}\}_{i=1}^n$, we show how an approximation F_n^* to F^* may be obtained from the data vector $g(t_{1n}), \ldots, g(t_{nn})$, using coefficients which are an intermediate step in the calculation of x_n.

In Section 2 we define F_n^* and show that $\lim_{n\to\infty} F_n^*(t) = F^*(t)$, $0 \leqslant t \leqslant 1$. In Section 3 we briefly mention some numerical results. In Section 4 we note how the results apply to more general linear operator equations. In Section 5 we note how the problem is formally equivalent to an optimal quadrature problem and compare it to other work.

2. THE ESTIMATE F_n^* OF F^*

Given an arbitrary mesh T_n of distinct points $\{t_{in}\}_{i=1}^n \equiv \{t_i\}_{i=1}^n$, then x_n, that element of minimal $\mathscr{H}_R$ norm satisfying the boundary conditions and $(L_m x_n)(t_i) = g(t_i)$, has a representation

$$x_n(t) = \sum_{\nu=0}^{m/2-1} d_{0\nu} R_{0\nu}(t) + \sum_{i=1}^{n} c_i \eta_{t_i}(t) + \sum_{\nu=0}^{m/2-1} d_{1\nu} R_{1\nu}(t), \tag{2.1}$$

where the $\{R_{i\nu}\}$ and η_{t_i} have been given in (1.4) and (1.5).

Forcing x_n to satisfy the boundary conditions and the differential equation at the mesh points leads to the following system of $n + m$ equations in the c's and d's:

$$\sum_{\nu=0}^{m/2-1} d_{0\nu} R_{0\nu}^{(\mu)}(0) + \sum_{i=0}^{n} c_i \eta_{t_i}^{(\mu)}(0) + \sum_{\nu=0}^{m/2-1} d_{1\nu} R_{1\nu}^{(\mu)}(0) = 0, \quad \mu = 0, 1,\ldots, m/2 - 1,$$

$$\sum_{\nu=0}^{m/2-1} d_{0\nu}(L_m R_{0\nu})(t_j) + \sum_{i=1}^{n} c_i(L_m \eta_{t_i})(t_j) + \sum_{\nu=0}^{m/2-1} d_{1\nu}(L_m R_{1\nu})(t_j) = g(t_j),$$
$$j = 1, 2,\ldots, n,$$

$$\sum_{\nu=0}^{m/2-1} d_{0\nu} R_{0\nu}^{(\mu)}(1) + \sum_{i=1}^{n} c_i \eta_{t_i}^{(\mu)}(1) + \sum_{\nu=0}^{m/2-1} d_{1\nu} R_{1\nu}^{(\mu)}(1) = 0,$$
$$\mu = 0, 1,\ldots, m/2 - 1. \tag{2.2}$$

These equations are equivalent to (2.1) and

$$\begin{aligned} \langle x_n, R_{0\nu}\rangle_R &= 0, & \nu &= 0, 1,\ldots, m/2 - 1, \\ \langle x_n, \eta_{t_i}\rangle_R &= g(t_i), & i &= 1, 2,\ldots, n, \\ \langle x_n, R_{1\nu}\rangle_R &= 0, & \nu &= 0, 1,\ldots, m/2 - 1. \end{aligned} \tag{2.3}$$

The system resulting from (2.2) is not particularly well suited for computation. When R is given by, e.g., (1.3), the solution can be expressed in terms of a B-spline basis, resulting in a linear system involving band matrices. See [2] for details of the calculation.

As an estimate F_n^* of F^* of (1.17) we take

$$F_n^*(t) = \int_0^t [\rho_n(s)\, \alpha(s)]^{1/(2q+1)}\, ds \Big/ \int_0^1 [\rho_n(s)\, \alpha(s)]^{1/(2q+1)}\, ds, \tag{2.4}$$

where ρ_n is an estimate of the ρ appearing in (1.10). The function ρ_n is formed from the vector $(c_1, c_2,\ldots, c_n)$ appearing in (2.2) as the piecewise linear function on $[0, 1]$ joining the points $(0, 0)$, $(t_1, c_1/h)$, $(t_2, c_2/h),\ldots, (t_n, c_n/h)$, $(1, 0)$, where $h = 1/(n + 1)$. Our main result is:

THEOREM. *Let α and ρ be strictly positive and continuous and* (*without loss of generality*) *let $t_i = i/(n + 1)$, $i = 1, 2,\ldots, n$. Then*

$$\lim_{n\to\infty} F_n^*(t) = F^*(t), \qquad t \in [0, 1].$$

Proof. We first show

$$\lim_{n\to\infty} \int_0^1 \rho_n(s)\, Q(t, s)\, ds = \int_0^1 \rho(s)\, Q(t, s)\, ds \equiv u(t), \qquad t \in [0, 1]. \tag{2.5}$$

Using the trapezoidal rule in each interval on the integral on the left-hand side we have

$$\begin{aligned}\int_0^1 \rho_n(s)\, Q(t,s)\, ds &= \sum_{i=1}^{n-1} [\{\rho_n(t_i)\, Q_t(t_i) + \rho_n(t_{i+1})\, Q_t(t_{i+1})\}\frac{h}{2} + O(h^2)] \\ &= \sum_{i=1}^{n} c_i Q_{t_i}(t) + \sum_{i=1}^{n-1} O(h^2).\end{aligned} \tag{2.6}$$

We will next show that $\| \sum_{i=1}^{n} c_i \eta_{t_i} - P_{V_n} y \|_R \to 0$, where y is the element in the decomposition

$$x = y + z, \qquad y \in V, \qquad z \in S_m .$$

This will guarantee that $\| \sum_{i=1}^{n} c_i Q_{t_i} - P_{T_n} u \|_Q \to 0$ by the isometric isomorphism, and, since $\| u - P_{T_n} u \|_Q \to 0$, it will follow that $\| \sum_{i=1}^{n} c_i Q_{t_i} - u \|_Q \to 0$, and hence $\sum_{i=1}^{n} c_i Q_{t_i}(t) \to u(t)$, and then (2.5) holds. Let $\gamma_1, ..., \gamma_m$ be an orthonormal basis for S_m. We may write

$$x = \sum_{i=1}^{n} \tilde{c}_i \eta_{t_i} + \delta + \sum_{i=1}^{m} \tilde{\theta}_i \gamma_i , \tag{2.7}$$

where

$$\sum_{i=1}^{n} \tilde{c}_i \eta_{t_i} = P_{V_n} y, \qquad \delta = (I - P_{V_n}) y. \tag{2.8}$$

Also, we may write

$$x_n = \sum_{i=1}^{n} c_i \eta_{t_i} + \sum_{j=1}^{m} \theta_j \gamma_j , \tag{2.9}$$

where $c = (c_1, ..., c_n)$ is as in (2.1) and (2.2). Letting Q_n be the $n \times n$ matrix with ijth entry $\langle \eta_{t_i}, \eta_{t_j} \rangle_R = (L_m \eta_{t_i})(t_j)$, and $\sum$ be the $n \times n$ matrix with ijth entry $\langle \eta_{t_i}, \gamma_j \rangle_R$, we have that (2.1) and (2.2) are equivalent to

$$x_n = (\eta_{t_1}, ..., \eta_{t_n}, \gamma_1, ..., \gamma_m) \left(\begin{array}{c|c} Q_n & \Sigma \\ \hline \Sigma' & I \end{array} \right)^{-1} \left(\begin{array}{c} \langle x, \eta_{t_1} \rangle_Q \\ \vdots \\ \langle x, \eta_{t_n} \rangle_Q \\ \hline \langle x, \gamma_1 \rangle_Q \\ \vdots \\ \langle x, \gamma_m \rangle_Q \end{array} \right). \tag{2.10}$$

Now, using (2.7), gives

$$\begin{pmatrix} \langle x, \eta_{t_1}\rangle_Q \\ \vdots \\ \langle x, \eta_{t_n}\rangle_Q \end{pmatrix} = Q_n\tilde{c} + \sum \tilde{\theta},$$

$$\begin{pmatrix} \langle x, \gamma_1\rangle_Q \\ \vdots \\ \langle x, \gamma_m\rangle_Q \end{pmatrix} = \sum{}' \tilde{c} + I\tilde{\theta} + \begin{pmatrix} \langle \delta, \gamma_1\rangle_Q \\ \vdots \\ \langle \delta, \gamma_m\rangle_Q \end{pmatrix}, \tag{2.11}$$

where $\tilde{c} = (\tilde{c}_1, ..., \tilde{c}_n)$, $\tilde{\theta} = (\tilde{\theta}_1, ..., \tilde{\theta}_m)$, and I is the $m \times m$ identity.

Substituting (2.11) into (2.10) gives

$$x_n = \sum_{i=1}^{n} \tilde{c}_i \eta_{t_i} + \sum_{j=1}^{m} \tilde{\theta}_j \gamma_j + P_S\delta,$$

where $P_S\delta$ is the projection of δ onto span($V_n \oplus S_m$). Since S_m may not be orthogonal to V_n, $P_S\delta$ will have a decomposition $P_S\delta = \delta_0 + \delta_1$, where $\delta_0 \in S_m$ and $\delta_1 \in V_n$. Thus

$$\sum_{i=1}^{n} c_i \eta_{t_i} - \sum_{i=1}^{n} \tilde{c}_i \eta_{t_i} \equiv \sum_{i=1}^{n} c_i \eta_{t_i} - P_{V_n} y = \delta_1,$$

and we want to show that $\| \delta_1 \|_R \to 0$. But the angle between V_n and S_m is bounded from below by the angle between V and S_m, and this entails the existence of a constant good for all n such that $\| \delta_1 \|_R \leqslant$ const $\| P_S\delta \|_R \leqslant$ const $\| y - P_{V_n} y \|_R \to 0$. Thus we have completed the proof of (2.5).

Equation (2.5) says $(\rho_n, Q_t)_{\mathscr{L}_2} \to (\rho, Q_t)_{\mathscr{L}_2}$, all $t \in [0, 1]$. Since Q is of full rank and $\mathscr{H}_Q$ is dense in $\mathscr{L}_2[0, 1]$, this entails that $\| \rho_n - \rho \|_{\mathscr{L}_2} \to 0$. It remains to show that $\rho_n \to \rho$ in $\mathscr{L}_2$ entails that

$$\int_0^t [\rho_n^2(u)\, \alpha(u)]^{1/(2q+1)}\, du \to \int_0^t [\rho^2(u)\, \alpha(u)]^{1/(2q+1)}\, du.$$

Letting $\rho = \rho_n + \epsilon_n$ and using the fact that

$$| \rho |^{2/(2q+1)} - | \epsilon_n |^{2/(2q+1)} \leqslant | \rho_n |^{2/(2q+1)} \leqslant | \rho |^{2/(2q+1)} + | \epsilon_n |^{2/(2q+1)}$$

and a Minkowski inequality gives the theorem.

We remark that we may compare the efficiency of a uniform mesh ($f(t) = 1$, $F(t) = t$) with the optimum mesh determined by F^*, $F^*(t) = \int_0^t f^*(s)\, ds$, by looking at the ratio

$$\left[\int_0^1 [\rho^2(s)\, \alpha(s)]^{1/2q}\, ds\right]^{2q+1} \Big/ \int_0^1 \rho^2(s)\, \alpha(s)\, ds.$$

Clearly, if $\rho^2(s)\,\alpha(s) = \text{const}$, this ratio is one, and the uniform mesh is optimum. The greater the difference between the "geometric" mean and the arithmetic mean of $\rho^2(s)\,\alpha(s)$, the greater the benefit of obtaining an optimum mesh.

The iterative determination of the mesh may be repeated any number of times, but the tradeoff between the cost of iteration and increased accuracy will depend on the problem. If data are determined experimentally and are costly to obtain then a multistage procedure becomes more attractive.

3. Numerical Results

Numerical results appear in [2]. For solving the problem the second time with the new mesh, an equivalent B-spline basis is used for S. See [2, 3]. The details of the B-spline formulation, considerations in the selection of $\mathscr{H}_R$ as well as other computational parameters and certain numerical comparisons may be found in [2]. A summary of numerical results for the problem

$$x''(t) + 10t\,x(t) = (-\pi^2 + 10t)\sin \pi t,$$
$$x(0) = x(1) = 0$$

is given in Table I. Since the actual solution $x(t) = \sin \pi t$ is known the maximum error using the optimized as well as a uniform mesh can be computed and are tabulated in Table I.

TABLE I

Comparison of Error Using Uniform and Approximately Optimum Mesh for the Test Problem

r	n	N	Approximately optimum mesh Error	Uniform mesh Error
5	10	15	0.35–5	0.31–4
		25	0.23–6	0.28–5
		35	0.50–7	0.57–6
		45	0.13–7	0.16–6
	20	15	0.63–6	0.31–4
		25	0.78–7	0.28–5
		35	0.12–7	0.57–6
		45	0.87–7	0.16–6
6	10	15	0.23–6	0.22–5
		25	0.67–8	0.91–7
	20	15	0.80–7	0.22–5
		25	0.17–7	0.91–7

4. Application to Design of Experiments for More General Linear Operator Equations

The iterative or sequential procedure for choosing a mesh clearly has more general application than to the solution of BVP's. Consider for example the Fredholm integral equation of the first kind

$$g(t) = \int_0^1 K(t, s)\, x(s)\, ds \tag{4.1}$$

arising in many experimental situations. If $x \in \mathscr{H}_R$, then $y \in \mathscr{H}_Q$ where $Q(s, t) = \int_0^1 \int_0^1 K(t, u)\, R(u, v)\, K(s, v)\, du dv$. (See [9].) If x_N is that element of minimal $\mathscr{H}_R$ norm satisfying $(Kx_N)(t) = g(t)$ for $t \in T_N$, and K is of full rank, then $\| x - x_N \|_R = \| g - P_{T_N} g \|_Q$ and the procedure for choosing T_N proceeds as before. However first kind equations are better solved by, e.g., regularization than by collocation for reasons noted by many people (see [13] for details). Provided that K is not "too" compact (as an operator from $\mathscr{L}_2$ to $\mathscr{L}_2$) and data points are expensive to obtain (as frequently happens in this context) an iterative procedure for choosing additional points may well turn out to be important. For a discussion of the experimental design problem when g is observed with noise, and regularization is used to solve (4.1), see [14].

5. Remarks

We note that the approach here contrasts with that of de Boor and Swartz [4]. They suppose x has $2q + m$ continuous derivatives (their k is our q), they use local piecewise polynomial approximating functions of degree $m + q - 1$ to approximate x by collocation and obtain pointwise $O(N^{-2q})$ convergence rates at certain special points when certain collocation points are zeros of the qth Legendre polynomial in subintervals. The approximations to x here are piecewise polynomials of degree $2r - 1 = 2q + 2m - 1$. Here pointwise convergence rates of $O(N^{-2q})$ hold uniformly over $[0, 1]$ for any mesh determined by a nice F and y satisfies (1.11), which is equivalent to $x \in W^{(2q+m)}$, and x satisfies some further boundary conditions. The convergence proof has been given in [10, Theorem 3]. The proof in [10] essentially uses $| x(t) - x_N(t)| = |\langle x - x_N, R_t - R_{tN}\rangle_R | \leqslant \| x - x_N \|_R \| R_t - R_{tN} \|_R$, where R_{tN} is the projection of R_t onto $S_m \oplus V_N$, and $\| x - x_N \|_R$ and $\| R_t - R_{tN} \|_R$ are each $O(N^{-q})$.

We note that this optimal mesh problem is formally an optimal quadrature problem, as follows. If

$$g(t) = \int_0^1 Q(t, s)\, \rho(s)\, ds$$

then g is a representer of weighted integration in $\mathscr{H}_Q$,

$$\langle g, u \rangle_Q = \int_0^1 \rho(s)\, u(s)\, ds, \qquad u \in \mathscr{H}_Q$$

Letting $P_{T_N} g = \sum_{i=1}^N w_i Q_{t_i}$ then a quadrature formula is obtained since

$$\langle P_{T_N} g, u \rangle_Q = \sum_{i=1}^N w_i \langle Q_{t_i}, u \rangle_Q = \sum_{i=1}^N w_i u(t_i).$$

An error bound is

$$\left| \int_0^1 \rho(s)\, u(s)\, ds - \sum_{i=1}^N w_i u(t_i) \right| = |\langle g - P_{T_N} g, u \rangle_Q| \leqslant \| g - P_{T_N} g \|_Q \, \| u \|_Q .$$

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